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## Positive solutions for Dirichlet problems of singular nonlinear fractional differential equations

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### ABSTRACT

In this paper, we investigate the existence of positive solutions for the singular fractional boundary value problem:  $D^\alpha u(t) + f(t, u(t), D^\mu u(t)) = 0$ ,  $u(0) = u(1) = 0$ , where  $1 < \alpha < 2$ ,  $0 < \mu \leq \alpha - 1$ ,  $D^\alpha$  is the standard Riemann–Liouville fractional derivative,  $f$  is a positive Carathéodory function and  $f(t, x, y)$  is singular at  $x = 0$ . By means of a fixed point theorem on a cone, the existence of positive solutions is obtained. The proofs are based on regularization and sequential techniques.

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### 1. Introduction

We are interested in the singular fractional Dirichlet boundary value problem

$$D^\alpha u(t) + f(t, u(t), D^\mu u(t)) = 0, \quad (1.1)$$

$$u(0) = u(1) = 0, \quad (1.2)$$

where  $1 < \alpha < 2$ ,  $\mu > 0$  are real numbers,  $\alpha - \mu \geq 1$ . Here  $f$  satisfies the Carathéodory conditions on  $[0, 1] \times \mathcal{B}$ ,  $\mathcal{B} = (0, \infty) \times \mathbb{R}$  ( $f \in \text{Car}([0, 1] \times \mathcal{B})$ ),  $f$  is positive,  $f(t, x, y)$  is singular at  $x = 0$  and  $D^\alpha$  is the standard Riemann–Liouville fractional derivative.

We recall that the Riemann–Liouville fractional derivative of order  $\beta > 0$  for a function  $v \in C(0, 1]$  is given by (see, e.g., [2,8,11])

$$D^\beta v(t) = \frac{1}{\Gamma(n - \beta)} \left( \frac{d}{dt} \right)^n \int_0^t (t - s)^{n - \beta - 1} v(s) ds$$

provided that the right hand side is pointwise defined on  $(0, 1]$ . Here  $n = [\beta] + 1$  and  $[\beta]$  means the integral part of the number  $\beta$ , and  $\Gamma$  is the Euler gamma function.

We say that  $f$  satisfies the Carathéodory conditions on the set  $[0, 1] \times \mathcal{B}$ ,  $\mathcal{B} = (0, \infty) \times \mathbb{R}$ , if

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- (i)  $f(\cdot, x, y) : [0, 1] \rightarrow \mathbb{R}$  is measurable for all  $(x, y) \in \mathcal{B}$ ,
- (ii)  $f(t, \cdot, \cdot) : \mathcal{B} \rightarrow \mathbb{R}$  is continuous for a.e.  $t \in [0, 1]$ ,
- (iii) for each compact set  $\mathcal{K} \subset \mathcal{B}$  there is a function  $m_{\mathcal{K}} \in L^1[0, 1]$  such that

$$|f(t, x, y)| \leq m_{\mathcal{K}}(t) \quad \text{for a.e. } t \in [0, 1] \text{ and all } (x, y) \in \mathcal{K}.$$

A function  $u \in C[0, 1]$  is called a *positive solution of problem (1.1), (1.2)* if  $u > 0$  on  $(0, 1)$ ,  $D^\mu u \in C[0, 1]$ ,  $D^\alpha u \in L^1[0, 1]$ ,  $u$  satisfies the boundary conditions (1.2) and equality (1.1) holds a.e. on  $[0, 1]$ .

Recently, fractional differential equations have been investigated extensively. The motivation for those works rises from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, chemistry, aerodynamics, electrodynamics of complex medium, and so on. For examples and details, see [1,5,6,8,10–12] and the references therein.

The Dirichlet problem for fractional differential equations and systems of fractional differential equations is discussed in papers [3,4,13–17]. In [3], the authors investigate the existence and multiplicity of positive solutions of problem

$$D^\alpha u(t) + h(t, u(t)) = 0,$$

$$u(0) = u(1) = 0,$$

where  $1 < \alpha \leq 2$  and  $h \in C([0, 1] \times [0, \infty))$  is nonnegative. The results are proved by means of the Krasnosel'skii fixed point theorem and the Leggett–Williams fixed point theorem. In [16], the system

$$D^{\alpha_1} u_1(t) + f_1(t, u_2(t), D^{\mu_1} u_2(t)) = 0,$$

$$\vdots$$

$$D^{\alpha_{n-1}} u_{n-1}(t) + f_{n-1}(t, u_n(t), D^{\mu_{n-1}} u_n(t)) = 0,$$

$$D^{\alpha_n} u_n(t) + f_n(t, u_1(t), D^{\mu_n} u_1(t)) = 0 \quad (1.3)$$

together with the boundary conditions

$$u_1(0) = u_2(0) = \cdots = u_n(0) = 0,$$

$$u_1(1) = u_2(1) = \cdots = u_n(1) = 0 \quad (1.4)$$

is studied. Here  $1 < \alpha_j < 2$ ,  $\mu_j > 0$ ,  $\alpha_j - \mu_{j-1} > 1$ ,  $j = 1, 2, \dots, n$ ,  $\mu_0 = \mu_n$ , and  $f_j \in C([0, 1] \times [0, \infty) \times \mathbb{R})$  is nonnegative. The existence and multiplicity results of positive solutions are proved by a nonlinear alternative of Leray–Schauder type and a fixed point theorem of Leggett–Williams type. The special case of problem (1.3), (1.4) for  $n = 2$  and  $f_1, f_2 \in C([0, 1] \times \mathbb{R}^2)$  is discussed in [14]. The existence result is proved by the Schauder fixed point theorem.

The Dirichlet boundary conditions

$$u(0) = v \neq 0, \quad u(1) = \rho \neq 0 \quad (1.5)$$

are used in papers [4] (here condition (1.5) is the special case of the nonlocal boundary conditions  $u(0) = g(u)$ ,  $u(1) = \rho$ ) and [13,17]. Since in (1.5) we have nonzero boundary values, the Riemann–Liouville fractional derivative  $D^\alpha$  is not suitable. Therefore, in [4,13,17] the fractional differential equation

$${}^C D^\alpha u(t) = h(t, u(t)), \quad 1 < \alpha \leq 2,$$

is investigated, where  ${}^C D^\alpha$  is the Caputo fractional derivative (see, e.g., [8])

$${}^C D^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds.$$

We observe that in [15], the Dirichlet problem  ${}^C D^\alpha u(t) = f(t, u(t), {}^C D^\mu u(t))$ ,  $u(0) = u(1) = 0$ , is considered for  $f \in C([0, 1] \times \mathbb{R}^2)$ , and the existence results are proved by the Schauder fixed point theorem.

No contributions exist, as far as we know, concerning the existence of positive solutions of problem (1.1), (1.2), where  $f$  is the Carathéodory function and  $f(t, x, y)$  is singular at the point  $x = 0$ . The aim of this paper is to discuss the existence of positive solution of such singular problems.

Throughout the paper  $\|x\| = \max\{|x(t)| : t \in [0, 1]\}$  is the norm in the space  $C[0, 1]$  while  $\|x\|_L = \int_0^1 |x(t)| dt$  is the norm in  $L^1[0, 1]$ , and  $1 < \alpha < 2$ ,  $0 < \mu \leq \alpha - 1$ .

We work with the following conditions on  $f$  in (1.1).

(H<sub>1</sub>)  $f \in \text{Car}([0, 1] \times \mathcal{B})$ ,  $\mathcal{B} = (0, \infty) \times \mathbb{R}$ ,

$$\lim_{x \rightarrow 0^+} f(t, x, y) = \infty \quad \text{for a.e. } t \in [0, 1] \text{ and all } y \in \mathbb{R}, \quad (1.6)$$

and there exists a positive constant  $m$  such that

$$f(t, x, y) \geq m(1-t)^{2-\alpha} \quad \text{for a.e. } t \in [0, 1] \text{ and all } (x, y) \in \mathcal{B}, \quad (1.7)$$

(H<sub>2</sub>)  $f$  fulfills the estimate

$$f(t, x, y) \leq \gamma(t)(q(x) + p(x) + \omega(|y|)) \quad \text{for a.e. } t \in [0, 1] \text{ and all } (x, y) \in \mathcal{B},$$

where  $\gamma \in L^1[0, 1]$ ,  $q \in C(0, \infty)$ ,  $p, \omega \in C[0, \infty)$  are positive,  $q$  is nonincreasing,  $p, \omega$  are nondecreasing,

$$\int_0^1 \gamma(t)q(Kt(1-t))dt < \infty, \quad K = \frac{m}{2\Gamma(\alpha-1)},$$

$$\lim_{x \rightarrow \infty} \frac{p(x) + \omega(x)}{x} = 0.$$

**Remark 1.1.** It follows from (1.6) that under condition (H<sub>2</sub>),  $\lim_{x \rightarrow 0^+} q(x) = \infty$ .

Since (1.1) is a singular equation we use regularization and sequential techniques for the existence of a positive solution of problem (1.1), (1.2). For this end, for each  $n \in \mathbb{N}$  define  $f_n$  by the formula

$$f_n(t, x, y) = \begin{cases} f(t, x, y) & \text{if } x \geq \frac{1}{n}, \\ f(t, \frac{1}{n}, y) & \text{if } 0 \leq x < \frac{1}{n}. \end{cases}$$

Then  $f_n \in \text{Car}([0, 1] \times \mathcal{B}_*)$ ,  $\mathcal{B}_* = [0, \infty) \times \mathbb{R}$ , and conditions (H<sub>1</sub>) and (H<sub>2</sub>) give

$$f_n(t, x, y) \geq m(1-t)^{2-\alpha} \quad \text{for a.e. } t \in [0, 1] \text{ and all } (x, y) \in \mathcal{B}_*, \quad (1.8)$$

$$\left. \begin{aligned} f_n(t, x, y) &\leq \gamma(t) \left( q\left(\frac{1}{n}\right) + p(x) + p(1) + \omega(|y|) \right) \\ &\text{for a.e. } t \in [0, 1] \text{ and all } (x, y) \in \mathcal{B}_*, \end{aligned} \right\} \quad (1.9)$$

$$\left. \begin{aligned} f_n(t, x, y) &\leq \gamma(t)(q(x) + p(x) + p(1) + \omega(|y|)) \\ &\text{for a.e. } t \in [0, 1] \text{ and all } (x, y) \in \mathcal{B}. \end{aligned} \right\} \quad (1.10)$$

We discuss the regular fractional differential equation

$$D^\alpha u(t) + f_n(t, u(t), D^\mu u(t)) = 0. \quad (1.11)$$

A function  $u \in C[0, 1]$  is a solution of (1.11) if  $D^\mu u \in C[0, 1]$ ,  $D^\alpha u \in L^1[0, 1]$  and (1.11) holds a.e. on  $[0, 1]$ .

The paper is organized as follows. In Section 2, we present some results of fractional calculus theory and auxiliary technical lemmas, which are used in the next two sections. Section 3 deals with the auxiliary regular problem (1.11), (1.2). We show that the solvability of this problem is reduced to the existence of a fixed point of an operator  $\mathcal{Q}_n$ . By a fixed point theorem of cone compression type due to Krasnosel'skii (see, e.g., [7,9]), the existence of a fixed point of  $\mathcal{Q}_n$  is proved. In Section 4, applying the results of Sections 2 and 3, we prove that problem (1.1), (1.2) has a positive solution. An example demonstrates the application of our results.

## 2. Preliminaries

The Riemann–Liouville fractional integral  $I^\beta v$  of  $v \in (0, 1] \rightarrow \mathbb{R}$  of order  $\beta > 0$  is defined by

$$I^\beta v(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s) ds$$

provided the right hand side is pointwise defined on  $(0, 1]$ . The following properties of the fractional calculus theory are well known (see, e.g., [8,11]).

- (i)  $D^\beta I^\beta v(t) = v(t)$  for a.e.  $t \in [0, 1]$ , where  $v \in L^1[0, 1]$ ,  $\beta > 0$ ,
- (ii)  $D^\beta v(t) = 0$  if and only if  $v(t) = \sum_{j=1}^n c_j t^{\beta-j}$ , where  $c_j$  ( $j = 1, \dots, n$ ) are arbitrary constants,  $n = [\beta] + 1$ ,  $\beta > 0$ ,

- (iii)  $I^\beta : C[0, 1] \rightarrow C[0, 1]$ ,  $I^\beta : L^1[0, 1] \rightarrow L^1[0, 1]$ ,  $\beta > 0$ ,  
 (iv)  $I^\beta I^\gamma v(t) = I^{\beta+\gamma} v(t)$  for  $t \in [0, 1]$ , where  $v \in L^1[0, 1]$ ,  $\beta + \gamma \geq 1$ .

The equality in property (iv) has the form of

$$\int_0^t (t-s)^{\beta-1} \left( \int_0^s (s-\tau)^{\gamma-1} v(\tau) d\tau \right) ds = B(\beta, \gamma) \int_0^t (t-s)^{\beta+\gamma-1} v(s) ds \quad (2.1)$$

for  $t \in [0, 1]$ , where  $v \in L^1[0, 1]$  and  $\beta + \gamma \geq 1$ . Here  $B$  is the beta function,  $B(\beta, \gamma) = \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$ .

Define the function  $G$  by the formula

$$G(t, s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} & \text{if } 0 \leq s \leq t \leq 1, \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)} & \text{if } 0 \leq t \leq s \leq 1. \end{cases} \quad (2.2)$$

The properties of the function  $G$  are collected in the following lemma.

**Lemma 2.1.** Let  $G$  be as in (2.2). Then

- (a)  $G \in C([0, 1] \times [0, 1])$ ,  
 (b)  $G(t, s) > 0$  for  $(t, s) \in (0, 1) \times (0, 1)$  and  $G(0, s) = 0 = G(1, s)$  for  $s \in [0, 1]$ ,  
 (c)  $\max\{G(t, s) : (t, s) \in [0, 1] \times [0, 1]\} = E$ , where  $E = \frac{1}{2^{2(\alpha-1)}\Gamma(\alpha)}$ .

**Proof.** It is easy to verify properties (a) and (b) of  $G$ . In view of the equality  $G(s, s) = \max\{G(t, s) : t \in [0, 1]\}$  for  $s \in [0, 1]$ , we have  $\max\{G(t, s) : (t, s) \in [0, 1] \times [0, 1]\} = \frac{1}{\Gamma(\alpha)} \max\{[s(1-s)]^{\alpha-1} : s \in [0, 1]\} = E$ .  $\square$

**Lemma 2.2.** Let  $g \in L_1[0, 1]$ . Then

$$u(t) = \int_0^1 G(t, s)g(s) ds$$

is the unique continuous solution on  $[0, 1]$  of the equation

$$D^\alpha u(t) + g(t) = 0$$

satisfying the boundary conditions (1.2).

**Proof.** We proceed as in the proof of Lemma 2.3 [3] (where  $g \in C[0, 1]$  is assumed) and use the properties of  $D^\beta$  and  $I^\beta$  given in (i)–(iv).  $\square$

**Lemma 2.3.** Let positive constants  $m$  and  $K$  be as in  $(H_1)$  and  $(H_2)$  and let  $r \in L^1[0, 1]$ ,  $r(t) \geq m(1-t)^{2-\alpha}$  for a.e.  $t \in [0, 1]$ . Then

$$\int_0^1 G(t, s)r(s) ds \geq Kt(1-t) \quad \text{for } t \in [0, 1]. \quad (2.3)$$

**Proof.** Let  $0 \leq s \leq t \leq 1$ . Then  $\Gamma(\alpha)G(t, s) = [t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}$ . Since  $t(1-s) > t-s$  for  $s \neq 0$  and  $t \neq 1$ , we deduce from the Lagrange mean value theorem that

$$[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1} = (\alpha-1)\xi^{\alpha-2}(1-t)s \geq (\alpha-1)[t(1-s)]^{\alpha-2}(1-t)s$$

for  $0 < s \leq t < 1$ , where  $t-s < \xi < t(1-s)$ . Hence

$$([t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1})(1-s)^{2-\alpha} \geq (\alpha-1)t^{\alpha-2}(1-t)s,$$

and consequently,

$$\int_0^t G(t, s)r(s) ds \geq \frac{(\alpha-1)m}{\Gamma(\alpha)} t^{\alpha-2}(1-t) \int_0^t s ds = \frac{(\alpha-1)m}{2\Gamma(\alpha)} t^\alpha(1-t). \quad (2.4)$$

Let  $0 \leq t \leq s \leq 1$ . Then

$$\begin{aligned} \int_t^1 G(t, s)r(s) ds &= \frac{1}{\Gamma(\alpha)} \int_t^1 [t(1-s)]^{\alpha-1} r(s) ds \geq \frac{mt^{\alpha-1}}{\Gamma(\alpha)} \int_t^1 (1-s) ds \\ &= \frac{mt^{\alpha-1}(1-t)^2}{2\Gamma(\alpha)}. \end{aligned} \quad (2.5)$$

Now, it follows from the equality

$$\int_0^1 G(t, s)r(s) ds = \int_0^t G(t, s)r(s) ds + \int_t^1 G(t, s)r(s) ds$$

and from (2.4) and (2.5) that

$$\begin{aligned} \int_0^1 G(t, s)r(s) ds &\geq \frac{m(1-t)}{2\Gamma(\alpha)} ((\alpha-1)t^\alpha + t^{\alpha-1}(1-t)) \\ &= \frac{m(1-t)}{2\Gamma(\alpha)} t^{\alpha-1} ((\alpha-2)t + 1) \geq \frac{m(\alpha-1)}{2\Gamma(\alpha)} t^{\alpha-1}(1-t) \\ &\geq \frac{m}{2\Gamma(\alpha-1)} t(1-t) \end{aligned}$$

for  $t \in [0, 1]$  because  $(\alpha-2)t + 1 \geq \alpha-1$  in this interval. Hence (2.3) is true.  $\square$

### 3. Auxiliary regular problem (1.11), (1.2)

Let us recall that  $1 < \alpha < 2$  and  $0 < \mu \leq \alpha-1$  throughout the paper. Let  $X = \{x \in C[0, 1]: D^\mu x \in C[0, 1]\}$ . Let  $X$  be equipped with the norm  $\|x\|_* = \max\{\|x\|, \|D^\mu x\|\}$ . Then  $X$  is a Banach space (see [14]). Define the cone  $P \subset X$  by

$$P = \{x \in X: x(t) \geq 0 \text{ for } t \in [0, 1]\}.$$

In order to prove that problem (1.11), (1.2) has a positive solution, we define an operator  $\mathcal{Q}_n$  on  $P$  by the formula

$$(\mathcal{Q}_n x)(t) = \int_0^1 G(t, s)f_n(s, x(s), D^\mu x(s)) ds.$$

The properties of the operator  $\mathcal{Q}_n$  are given in the following lemma.

**Lemma 3.1.** *Let  $(H_1)$  and  $(H_2)$  hold. Then  $\mathcal{Q}_n : P \rightarrow P$  and  $\mathcal{Q}_n$  is a completely continuous operator.*

**Proof.** Let  $x \in P$  and let  $\rho(t) = f_n(t, x(t), D^\mu x(t))$  for a.e.  $t \in [0, 1]$ . Then  $\rho \in L^1[0, 1]$  because  $f_n \in \text{Car}([0, 1] \times \mathcal{B}_*)$ , and  $\rho$  is positive. It follows from the equality

$$\begin{aligned} (\mathcal{Q}_n x)(t) &= \int_0^1 G(t, s)\rho(s) ds \\ &= \frac{1}{\Gamma(\alpha)} \left[ t^{\alpha-1} \int_0^1 (1-s)^{\alpha-1} \rho(s) ds - \int_0^t (t-s)^{\alpha-1} \rho(s) ds \right], \end{aligned}$$

from  $\int_0^t (t-s)^{\alpha-1} \rho(s) ds \in C[0, 1]$  and from  $G \geq 0$  by Lemma 2.1(b) that  $\mathcal{Q}_n x \in C[0, 1]$  and  $(\mathcal{Q}_n x)(t) \geq 0$  for  $t \in [0, 1]$ . Next, using the equality

$$\int_0^t (t-s)^{-\mu} s^{\alpha-1} ds = t^{\alpha-\mu} B(\alpha, 1-\mu), \quad t \in [0, 1],$$

we have (cf. (2.1))

$$\begin{aligned}
(D^\mu \mathcal{Q}_n x)(t) &= \frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_0^t (t-s)^{-\mu} (\mathcal{Q}_n x)(s) ds \\
&= \frac{1}{\Gamma(1-\mu)} \frac{d}{dt} \int_0^t (t-s)^{-\mu} \left( \int_0^1 G(s, \tau) \rho(\tau) d\tau \right) ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\mu)} \frac{d}{dt} \int_0^t (t-s)^{-\mu} \left( s^{\alpha-1} \int_0^1 (1-\tau)^{\alpha-1} \rho(\tau) d\tau - \int_0^s (s-\tau)^{\alpha-1} \rho(\tau) d\tau \right) ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\mu)} \frac{d}{dt} \left( \int_0^1 (1-\tau)^{\alpha-1} \rho(\tau) d\tau \cdot \int_0^t (t-s)^{-\mu} s^{\alpha-1} ds \right. \\
&\quad \left. - B(\alpha, 1-\mu) \int_0^t (t-\tau)^{\alpha-\mu} \rho(\tau) d\tau \right) \\
&= \frac{B(\alpha, 1-\mu)}{\Gamma(\alpha)\Gamma(1-\mu)} \frac{d}{dt} \left( t^{\alpha-\mu} \int_0^1 (1-\tau)^{\alpha-1} \rho(\tau) d\tau - \int_0^t (t-\tau)^{\alpha-\mu} \rho(\tau) d\tau \right) \\
&= \frac{(\alpha-\mu)B(\alpha, 1-\mu)}{\Gamma(\alpha)\Gamma(1-\mu)} \left( t^{\alpha-\mu-1} \int_0^1 (1-\tau)^{\alpha-1} \rho(\tau) d\tau - \int_0^t (t-\tau)^{\alpha-\mu-1} \rho(\tau) d\tau \right).
\end{aligned}$$

In particular,

$$(D^\mu \mathcal{Q}_n x)(t) = \frac{1}{\Gamma(\alpha-\mu)} \left( t^{\alpha-\mu-1} \int_0^1 (1-s)^{\alpha-1} \rho(s) ds - \int_0^t (t-s)^{\alpha-\mu-1} \rho(s) ds \right) \quad (3.1)$$

since  $B(\alpha, 1-\mu) = \frac{\Gamma(\alpha)\Gamma(1-\mu)}{(\alpha-\mu)\Gamma(\alpha-\mu)}$ . Hence  $D^\mu \mathcal{Q}_n x \in C[0, 1]$ . To summarize,  $\mathcal{Q}_n : P \rightarrow P$ .

In order to prove that  $\mathcal{Q}_n$  is a continuous operator, let  $\{x_m\} \subset P$  be a convergent sequence and let  $\lim_{m \rightarrow \infty} \|x_m - x\|_* = 0$ . Then  $x \in P$  and  $\|x_m\|_* \leq S$  for  $m \in \mathbb{N}$ , where  $S$  is a positive constant. Keeping in mind that  $f_n \in \text{Car}([0, 1] \times \mathcal{B}_*)$ , we have

$$\lim_{m \rightarrow \infty} f_n(t, x_m(t), D^\mu x_m(t)) = f_n(t, x(t), D^\mu x(t)) \quad \text{for a.e. } t \in [0, 1].$$

Since, by (1.8) and (1.9),

$$0 < f_n(t, x_m(t), D^\mu x_m(t)) \leq \gamma(t) \left( q \left( \frac{1}{n} \right) + p(S) + p(1) + \omega(S) \right),$$

the Lebesgue dominated convergence theorem gives

$$\lim_{m \rightarrow \infty} \int_0^1 |f_n(t, x_m(t), D^\mu x_m(t)) - f_n(t, x(t), D^\mu x(t))| dt = 0. \quad (3.2)$$

Now we deduce from (3.2), Lemma 2.1(c) and from the inequalities (cf. (3.1))

$$\begin{aligned}
|(\mathcal{Q}_n x_m)(t) - (\mathcal{Q}_n x)(t)| &\leq E \int_0^1 |f_n(s, x_m(s), D^\mu x_m(s)) - f(s, x(s), D^\mu x(s))| ds, \\
\Gamma(\alpha-\mu) |(D^\mu \mathcal{Q}_n x_m)(t) - (D^\mu \mathcal{Q}_n x)(t)| \\
&= \left| t^{\alpha-\mu-1} \int_0^1 (1-s)^{\alpha-1} (f_n(s, x_m(s), D^\mu x_m(s)) - f_n(s, x(s), D^\mu x(s))) ds \right. \\
&\quad \left. - \int_0^t (t-s)^{\alpha-\mu-1} (f_n(s, x_m(s), D^\mu x_m(s)) - f_n(s, x(s), D^\mu x(s))) ds \right|
\end{aligned}$$

$$\leq 2 \int_0^1 |f_n(s, x_m(s), D^\mu x_m(s)) - f(s, x(s), D^\mu x(s))| ds$$

that  $\lim_{m \rightarrow \infty} \|\mathcal{Q}_n x_m - \mathcal{Q}_n x\|_* = 0$ , which proves that  $\mathcal{Q}_n$  is a continuous operator. Finally, let  $\Omega \subset P$  be bounded in  $X$  and let  $\|x\|_* \leq L$  for all  $x \in \Omega$ , where  $L$  is a positive constant. In view of  $f_n \in \text{Car}([0, 1] \times \mathcal{B}_*)$ , there exists  $\varphi \in L^1[0, 1]$  such that

$$0 < f_n(t, x(t), D^\mu x(t)) \leq \varphi(t) \quad \text{for a.e. } t \in [0, 1] \text{ and all } x \in \Omega. \quad (3.3)$$

Then

$$|(\mathcal{Q}_n x)(t)| \leq E \int_0^1 f_n(s, x(s), D^\mu x(s)) ds \leq E \|\varphi\|_L$$

and (cf. (3.1))

$$\begin{aligned} |(D^\mu \mathcal{Q}_n x)(t)| &\leq \frac{1}{\Gamma(\alpha - \mu)} \left| t^{\alpha - \mu - 1} \int_0^1 (1-s)^{\alpha-1} f_n(s, x(s), D^\mu x(s)) ds - \int_0^t (t-s)^{\alpha - \mu - 1} f_n(s, x(s), D^\mu x(s)) ds \right| \\ &\leq \frac{2}{\Gamma(\alpha - \mu)} \|\varphi\|_L \end{aligned}$$

for  $t \in [0, 1]$  and  $x \in \Omega$ . Hence  $\|\mathcal{Q}_n x\| \leq E \|\varphi\|_L$  and  $\|D^\mu \mathcal{Q}_n x\| \leq \frac{2}{\Gamma(\alpha - \mu)} \|\varphi\|_L$  for  $x \in \Omega$ , and so  $\mathcal{Q}_n(\Omega)$  is bounded in  $X$ . Let  $0 \leq t_1 < t_2 \leq 1$ . Then (cf. (3.3))

$$\begin{aligned} &|(\mathcal{Q}_n x)(t_2) - (\mathcal{Q}_n x)(t_1)| \\ &= \frac{1}{\Gamma(\alpha)} \left| \int_0^1 ([t_2(1-s)]^{\alpha-1} - [t_1(1-s)]^{\alpha-1}) f_n(s, x(s), D^\mu x(s)) ds \right. \\ &\quad \left. + \int_0^{t_1} ((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}) f_n(s, x(s), D^\mu x(s)) ds - \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f_n(s, x(s), D^\mu x(s)) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \left( \int_0^1 (t_2^{\alpha-1} - t_1^{\alpha-1})(1-s)^{\alpha-1} \varphi(s) ds + \int_0^{t_1} ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}) \varphi(s) ds \right. \\ &\quad \left. + (t_2 - t_1)^{\alpha-1} \int_{t_1}^{t_2} \varphi(s) ds \right) \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} &|(D^\mu \mathcal{Q}_n x)(t_2) - (D^\mu \mathcal{Q}_n x)(t_1)| \\ &= \frac{1}{\Gamma(\alpha - \mu)} \left| (t_2^{\alpha - \mu - 1} - t_1^{\alpha - \mu - 1}) \int_0^1 (1-s)^{\alpha-1} f_n(s, x(s), D^\mu x(s)) ds \right. \\ &\quad \left. - \int_0^{t_2} (t_2-s)^{\alpha - \mu - 1} f_n(s, x(s), D^\mu x(s)) ds + \int_0^{t_1} (t_1-s)^{\alpha - \mu - 1} f_n(s, x(s), D^\mu x(s)) ds \right| \\ &\leq \frac{1}{\Gamma(\alpha - \mu)} \left( (t_2^{\alpha - \mu - 1} - t_1^{\alpha - \mu - 1}) \|\varphi\|_L + \int_0^{t_1} ((t_2-s)^{\alpha - \mu - 1} - (t_1-s)^{\alpha - \mu - 1}) \varphi(s) ds \right. \\ &\quad \left. + (t_2 - t_1)^{\alpha - \mu - 1} \int_{t_1}^{t_2} \varphi(s) ds \right) \quad \text{if } \alpha > \mu + 1, \end{aligned} \quad (3.5)$$

$$\left| (D^\mu \mathcal{Q}_n x)(t_2) - (D^\mu \mathcal{Q}_n x)(t_1) \right| \leq \int_{t_1}^{t_2} \varphi(s) ds \quad \text{if } \alpha = \mu + 1. \quad (3.6)$$

Let us choose an arbitrary  $\varepsilon > 0$ . Since the functions  $t^{\alpha-1}$ ,  $t^{\alpha-\mu-1}$  are uniformly continuous on  $[0, 1]$  and  $|t-s|^{\alpha-1}$ ,  $|t-s|^{\alpha-\mu-1}$  on  $[0, 1] \times [0, 1]$ , there exists  $\delta > 0$  such that for each  $0 \leq t_1 < t_2 \leq 1$ ,  $t_2 - t_1 < \delta$ ,  $0 \leq s \leq t_1$ , we have  $t_2^{\alpha-1} - t_1^{\alpha-1} < \varepsilon$ ,  $t_2^{\alpha-\mu-1} - t_1^{\alpha-\mu-1} < \varepsilon$ ,  $0 < (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} < \varepsilon$ ,  $0 < (t_2 - s)^{\alpha-\mu-1} - (t_1 - s)^{\alpha-\mu-1} < \varepsilon$ . Then, for  $x \in \Omega$  and  $0 \leq t_1 < t_2 \leq 1$ ,  $t_2 - t_1 < \min\{\delta, \varepsilon^{-1/\alpha}\}$ , we conclude from inequalities (3.4)–(3.6) that the inequalities

$$\begin{aligned} \left| (\mathcal{Q}_n x)(t_2) - (\mathcal{Q}_n x)(t_1) \right| &< \frac{3\varepsilon}{\Gamma(\alpha)} \|\varphi\|_L, \\ \left| (D^\mu \mathcal{Q}_n x)(t_2) - (D^\mu \mathcal{Q}_n x)(t_1) \right| &\leq \begin{cases} \frac{3\varepsilon}{\Gamma(\alpha-\mu)} \|\varphi\|_L & \text{if } \alpha > \mu + 1, \\ \int_{t_1}^{t_2} \varphi(s) ds & \text{if } \alpha = \mu + 1 \end{cases} \end{aligned}$$

hold. Hence the sets of functions  $\mathcal{Q}_n(\Omega)$  and  $\{D^\mu \mathcal{Q}_n x : x \in \Omega\}$  are bounded in  $C[0, 1]$  and equicontinuous on  $[0, 1]$ . Consequently,  $\mathcal{Q}_n(\Omega)$  is relatively compact in  $X$  by the Arzelà–Ascoli theorem. We have proved that  $\mathcal{Q}_n$  is a completely continuous operator.  $\square$

The next result follows immediately from Lemma 2.2.

**Lemma 3.2.** *Let  $(H_1)$  and  $(H_2)$  hold. Then any fixed point of the operator  $\mathcal{Q}_n$  is a solution of problem (1.11), (1.2).*

The following fixed point result of cone compression type due to Krasnosel'skii is fundamental for the solvability of problem (1.11), (1.2).

**Lemma 3.3.** *(See, e.g., [7,9].) Let  $Y$  be a Banach space, and  $P \subset Y$  be a cone in  $Y$ . Let  $\Omega_1, \Omega_2$  be bounded open balls of  $Y$  centered at the origin with  $\overline{\Omega}_1 \subset \Omega_2$ . Suppose that  $\mathcal{A} : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow P$  is a completely continuous operator such that*

$$\|\mathcal{A}x\| \geq \|x\| \quad \text{for } x \in P \cap \partial\Omega_1, \quad \|\mathcal{A}x\| \leq \|x\| \quad \text{for } x \in P \cap \partial\Omega_2$$

*hold. Then  $\mathcal{A}$  has a fixed point in  $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ .*

We are now in a position to give the existence result for the regular problem (1.11), (1.2).

**Lemma 3.4.** *Let  $(H_1)$  and  $(H_2)$  hold. Then problem (1.11), (1.2) has a solution.*

**Proof.** By Lemmas 3.1 and 3.2,  $\mathcal{Q}_n : P \rightarrow P$  is completely continuous and  $u$  is a solution of problem (1.11), (1.2) if  $u$  solves the operator equation  $u = \mathcal{Q}_n u$ . In order to apply Lemma 3.3, we separate the proof into two steps.

Step 1. Let

$$\Omega_1 = \left\{ x \in X : \|x\|_* < \frac{K}{4} \right\},$$

where  $K$  is as in  $(H_2)$ . It follows from Lemma 2.3 and from the definition of  $\mathcal{Q}_n$  that  $(\mathcal{Q}_n x)(t) \geq Kt(1-t)$  for  $t \in [0, 1]$  and  $x \in P$ . Hence  $\|\mathcal{Q}_n x\| \geq \frac{K}{4}$  for  $x \in P$ , and consequently,

$$\|\mathcal{Q}_n x\|_* \geq \|x\|_* \quad \text{for } x \in P \cap \partial\Omega_1. \quad (3.7)$$

Step 2. Inequality (1.9) and Lemma 2.1(c) imply that for  $x \in P$ ,

$$\begin{aligned} \left| \mathcal{Q}_n x(t) \right| &\leq E \int_0^1 \gamma(s) \left( q\left(\frac{1}{n}\right) + p(x(s)) + p(1) + \omega(|D^\mu x(s)|) \right) ds \\ &\leq E \left( q\left(\frac{1}{n}\right) + p(\|x\|) + p(1) + \omega(\|D^\mu x\|) \right) \|\gamma\|_L \end{aligned}$$

and (cf. (3.1) for  $\rho(t) = f_n(t, x(t), D^\mu x(t))$ )

$$\left| (D^\mu \mathcal{Q}_n x)(t) \right| \leq \frac{2}{\Gamma(\alpha - \mu)} \left( q\left(\frac{1}{n}\right) + p(\|x\|) + p(1) + \omega(\|D^\mu x\|) \right) \|\gamma\|_L$$

because  $p, \omega$  are nondecreasing by  $(H_2)$ . Let  $M = \max\{E, \frac{2}{\Gamma(\alpha - \mu)}\}$ . Hence for  $x \in P$ , the inequality



$$\|\mathcal{Q}_n x\|_* \leq M \left( q \left( \frac{1}{n} \right) + p(\|x\|_*) + p(1) + \omega(\|x\|_*) \right) \|\gamma\|_L \quad (3.8)$$

is fulfilled. Since  $\lim_{v \rightarrow \infty} \frac{p(v) + \omega(v)}{v} = 0$  by  $(H_2)$ , there exists  $S > 0$  such that

$$M \left( q \left( \frac{1}{n} \right) + p(S) + p(1) + \omega(S) \right) \|\gamma\|_L \leq S. \quad (3.9)$$

Let

$$\Omega_2 = \{x \in X: \|x\|_* < S\}.$$

Then (cf. (3.8) and (3.9))

$$\|\mathcal{Q}_n x\|_* \leq \|x\|_* \quad \text{for } x \in P \cap \partial\Omega_2. \quad (3.10)$$

Applying Lemma 3.3 for  $Y = X$  and  $\mathcal{A} = \mathcal{Q}_n$ , we conclude from (3.7) and (3.10) that  $\mathcal{Q}_n$  has a fixed point in  $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ . Consequently, problem (1.11), (1.2) has a solution by Lemma 3.2.  $\square$

The following lemma deals with sequences of solutions to problem (1.11), (1.2).

**Lemma 3.5.** *Let  $(H_1)$  and  $(H_2)$  hold. Let  $u_n$  be a solution of problem (1.11), (1.2). Then the sequences  $\{u_n\}$  and  $\{D^\mu u_n\}$  are relatively compact in  $C[0, 1]$ .*

**Proof.** We note that

$$u_n(t) = \int_0^1 G(t, s) f_n(s, u_n(s), D^\mu u_n(s)) \, ds, \quad t \in [0, 1], \quad n \in \mathbb{N}, \quad (3.11)$$

and (cf. (3.1) with  $\rho = f_n(t, u_n(t), D^\mu u_n(t))$ )

$$\begin{aligned} D^\mu u_n(t) &= \frac{1}{\Gamma(\alpha - \mu)} \left( t^{\alpha - \mu - 1} \int_0^1 (1 - s)^{\alpha - 1} f_n(s, u_n(s), D^\mu u_n(s)) \, ds \right. \\ &\quad \left. - \int_0^t (t - s)^{\alpha - \mu - 1} f_n(s, u_n(s), D^\mu u_n(s)) \, ds \right) \end{aligned} \quad (3.12)$$

for  $t \in [0, 1]$  and  $n \in \mathbb{N}$ . It follows from (1.8), (3.11) and Lemma 2.3 that

$$u_n(t) \geq Kt(1 - t) \quad \text{for } t \in [0, 1], \quad n \in \mathbb{N}. \quad (3.13)$$

Therefore (cf. (1.10)),

$$f_n(t, u_n(t), D^\mu u_n(t)) \leq \gamma(t) [q(Kt(1 - t)) + p(u_n(t)) + p(1) + \omega(|D^\mu u_n(t)|)] \quad (3.14)$$

for a.e.  $t \in [0, 1]$  and all  $n \in \mathbb{N}$ . Now, by (3.11), (3.12) and Lemma 2.1(c),

$$\begin{aligned} u_n(t) &\leq E[W + (p(\|u_n\|) + p(1) + \omega(\|D^\mu u_n\|)) \|\gamma\|_L], \\ |D^\mu u_n(t)| &\leq \frac{2}{\Gamma(\alpha - \mu)} [W + (p(\|u_n\|) + p(1) + \omega(\|D^\mu u_n\|)) \|\gamma\|_L] \end{aligned}$$

for  $t \in [0, 1]$  and  $n \in \mathbb{N}$ , where

$$W = \int_0^1 \gamma(t) q(Kt(1 - t)) \, dt < \infty.$$

In particular,

$$\|u_n\|_* \leq \max \left\{ E, \frac{2}{\Gamma(\alpha - \mu)} \right\} [W + (p(\|u_n\|_*) + p(1) + \omega(\|u_n\|_*)) \|\gamma\|_L] \quad (3.15)$$

for  $n \in \mathbb{N}$ . Since  $\lim_{v \rightarrow \infty} \frac{p(v) + \omega(v)}{v} = 0$ , there exists  $S > 0$  such that

$$\max \left\{ E, \frac{2}{\Gamma(\alpha - \mu)} \right\} (W + (p(v) + p(1) + \omega(v)) \|\gamma'\|_L) < v \quad \text{for each } v \geq S.$$

Therefore (cf. (3.15)),

$$\|u_n\|_* < S \quad \text{for } n \in \mathbb{N}. \quad (3.16)$$

Hence the sequences  $\{u_n\}$  and  $\{D^\mu u_n\}$  are bounded in  $C[0, 1]$ .

Now, we prove that  $\{u_n\}$  and  $\{D^\mu u_n\}$  are equicontinuous on  $[0, 1]$ . Let  $0 \leq t_1 < t_2 \leq 1$ . Then

$$\begin{aligned} |u_n(t_2) - u_n(t_1)| &= \left| \int_0^1 (G(t_2, s) - G(t_1, s)) f_n(s, u_n(s), D^\mu u_n(s)) ds \right| \\ &= \frac{1}{\Gamma(\alpha)} \left| (t_2^{\alpha-1} - t_1^{\alpha-1}) \int_0^1 (1-s)^{\alpha-1} f_n(s, u_n(s), D^\mu u_n(s)) ds \right. \\ &\quad \left. - \int_0^{t_1} ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}) f_n(s, u_n(s), D^\mu u_n(s)) ds \right. \\ &\quad \left. - \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f_n(s, u_n(s), D^\mu u_n(s)) ds \right| \end{aligned} \quad (3.17)$$

and (cf. (3.1))

$$\begin{aligned} |D^\mu u_n(t_2) - D^\mu u_n(t_1)| &= |(D^\mu \mathcal{Q}_n u_n)(t_2) - (D^\mu \mathcal{Q}_n u_n)(t_1)| \\ &= \frac{1}{\Gamma(\alpha - \mu)} \left| (t_2^{\alpha-\mu-1} - t_1^{\alpha-\mu-1}) \int_0^1 (1-s)^{\alpha-1} f_n(s, u_n(s), D^\mu u_n(s)) ds \right. \\ &\quad \left. - \int_0^{t_1} ((t_2-s)^{\alpha-\mu-1} - (t_1-s)^{\alpha-\mu-1}) f_n(s, u_n(s), D^\mu u_n(s)) ds \right. \\ &\quad \left. - \int_{t_1}^{t_2} (t_2-s)^{\alpha-\mu-1} f_n(s, u_n(s), D^\mu u_n(s)) ds \right| \quad \text{if } \alpha > \mu + 1, \end{aligned} \quad (3.18)$$

$$|D^\mu u_n(t_2) - D^\mu u_n(t_1)| = \left| \int_{t_1}^{t_2} f_n(s, u_n(s), D^\mu u_n(s)) ds \right| \quad \text{if } \alpha = \mu + 1. \quad (3.19)$$

We proceed analogously to the proof of Lemma 3.1. Let us choose an arbitrary  $\varepsilon > 0$ . Then there exists  $\delta_0 > 0$  such that for each  $0 \leq t_1 < t_2 \leq 1$ ,  $t_2 - t_1 < \delta_0$  and  $0 \leq s \leq t_1$ , we have  $t_2^{\alpha-1} - t_1^{\alpha-1} < \varepsilon$ ,  $t_2^{\alpha-\mu-1} - t_1^{\alpha-\mu-1} < \varepsilon$ ,  $(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} < \varepsilon$ ,  $(t_2-s)^{\alpha-\mu-1} - (t_1-s)^{\alpha-\mu-1} < \varepsilon$ . Let  $0 < \delta < \min\{\delta_0, \frac{\alpha-1}{\sqrt{\varepsilon}}\}$ . Now, using the inequality (cf. (3.14) and (3.16))

$$0 < f_n(t, u_n(t), D^\mu u_n(t)) \leq \gamma(t)(q(Kt(1-t)) + p(S) + p(1) + \omega(S))$$

for a.e.  $t \in [0, 1]$  and all  $n \in \mathbb{N}$ , we conclude from (3.17)–(3.19) that for  $0 \leq t_1 < t_2 \leq 1$ ,  $t_2 - t_1 < \delta$  and  $n \in \mathbb{N}$ , the following inequalities are fulfilled:

$$\begin{aligned} |u_n(t_2) - u_n(t_1)| &\leq \frac{\varepsilon}{\Gamma(\alpha)} \left( \int_0^1 (1-t)^{\alpha-1} \gamma(t)(q(Kt(1-t)) + p(S) + p(1) + \omega(S)) dt \right. \\ &\quad \left. + \int_0^{t_2} \gamma(t)(q(Kt(1-t)) + p(S) + p(1) + \omega(S)) dt \right) \end{aligned}$$

$$< \frac{2\varepsilon}{\Gamma(\alpha)} \int_0^1 \gamma(t)(q(Kt(1-t)) + p(S) + p(1) + \omega(S)) dt,$$

and

$$\begin{aligned} & |D^\mu u_n(t_2) - D^\mu u_n(t_1)| \\ & \leq \frac{\varepsilon}{\Gamma(\alpha - \mu)} \left( \int_0^1 (1-t)^{\alpha-1} \gamma(t)(q(Kt(1-t)) + p(S) + p(1) + \omega(S)) dt \right. \\ & \quad \left. + \int_0^{t_2} \gamma(t)(q(Kt(1-t)) + p(S) + p(1) + \omega(S)) dt \right) \\ & < \frac{2\varepsilon}{\Gamma(\alpha - \mu)} \int_0^1 \gamma(t)(q(Kt(1-t)) + p(S) + p(1) + \omega(S)) dt \end{aligned}$$

provided  $\alpha > \mu + 1$ ,

$$|D^\mu u_n(t_2) - D^\mu u_n(t_1)| \leq \int_{t_1}^{t_2} \gamma(t)(q(Kt(1-t)) + p(S) + p(1) + \omega(S)) dt$$

provided  $\alpha = \mu + 1$ . As a result,  $\{u_n\}$  and  $\{D^\mu u_n\}$  are equicontinuous on  $[0, 1]$ . Hence  $\{u_n\}$  and  $\{D^\mu u_n\}$  are relatively compact in  $C[0, 1]$  by the Arzelà–Ascoli theorem.  $\square$

#### 4. Positive solutions of problem (1.1), (1.2) and an example

**Theorem 4.1.** Let  $(H_1)$  and  $(H_2)$  hold. Then problem (1.1), (1.2) has a positive solution  $u$  and

$$u(t) \geq Kt(1-t) \quad \text{for } t \in [0, 1]. \quad (4.1)$$

**Proof.** Lemmas 3.4 and 3.5 guarantee that problem (1.1), (1.2) has a solution  $u_n$  satisfying (3.13) and  $\{u_n\}$ ,  $\{D^\mu u_n\}$  are relatively compact in  $C[0, 1]$ . Hence  $\{u_n\}$  is relatively compact in  $X$ , and therefore, there exist  $u \in X$  and a subsequence  $\{u_{k_n}\}$  of  $\{u_n\}$  such that  $\lim_{n \rightarrow \infty} u_{k_n} = u$  in  $X$ . Consequently,  $u \in P$ ,  $u$  satisfies (4.1) and

$$\lim_{n \rightarrow \infty} f_{k_n}(t, u_{k_n}(t), D^\mu u_{k_n}(t)) = f(t, u(t), D^\mu u(t)) \quad \text{for a.e. } t \in [0, 1].$$

Since  $\{u_n\}$  fulfills (3.16), where  $S$  is a positive constant, it follows from inequalities (1.10) and (3.13) and from Lemma 2.1(c) that the inequality

$$0 \leq G(t, s) f_n(s, u_n(s), D^\mu u_n(s)) \leq E \gamma(s)(q(Ks(1-s)) + p(S) + p(1) + \omega(S))$$

holds for a.e.  $s \in [0, 1]$  and all  $t \in [0, 1]$ ,  $n \in \mathbb{N}$ . Hence

$$\lim_{n \rightarrow \infty} \int_0^1 G(t, s) f_{k_n}(s, u_{k_n}(s), D^\mu u_{k_n}(s)) ds = \int_0^1 G(t, s) f(s, u(s), D^\mu u(s)) ds$$

for  $t \in [0, 1]$  by the Lebesgue dominated convergence theorem. Now, passing to the limit as  $n \rightarrow \infty$  in

$$u_{k_n}(t) = \int_0^1 G(t, s) f_{k_n}(s, u_{k_n}(s), D^\mu u_{k_n}(s)) ds$$

we have

$$u(t) = \int_0^1 G(t, s) f(s, u(s), D^\mu u(s)) ds \quad \text{for } t \in [0, 1]. \quad (4.2)$$

Consequently,  $u$  is a positive solution of problem (1.1), (1.2) by Lemma 2.2.  $\square$

**Example 4.2.** Let  $\beta, \nu, \rho \in (0, 1)$ . Then the function

$$f(t, x, y) = \frac{1}{\sqrt{|2t-1|}} \left( x^\beta + \frac{1}{x^\nu} + |y|^\rho \right)$$

satisfies condition  $(H_1)$  because  $f(t, x, y) \geq \frac{1}{\sqrt{|2t-1|}} \geq 1$  for  $t \in [0, 1] \setminus \{\frac{1}{2}\}$ ,  $(x, y) \in (0, \infty) \times \mathbb{R}$ , and condition  $(H_2)$  for  $\gamma(t) = \frac{1}{\sqrt{|2t-1|}}$ ,  $q(x) = \frac{1}{x^\nu}$ ,  $p(x) = x^\beta$ ,  $\omega(y) = y^\rho$  and  $K = \frac{1}{2\Gamma(\alpha-1)}$ . Hence Theorem 4.1 guarantees that for each  $\alpha \in (1, 2)$  and each  $\mu > 0$  such that  $\alpha - \mu \geq 1$ , the fractional differential equation

$$D^\alpha u(t) = \frac{1}{\sqrt{|2t-1|}} \left( (u(t))^\beta + \frac{1}{(u(t))^\nu} + |D^\mu u(t)|^\rho \right)$$

has a positive solution  $u$  satisfying the boundary condition (1.2) and  $u(t) \geq \frac{1}{2\Gamma(\alpha-1)} t(1-t)$  for  $t \in [0, 1]$ .

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